2. cvičení - řešení

Příklad 1 (a)

$$\int_0^1 (x + x^3 + 2x^7)\,\mathrm{d}x = \left[\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^8}{4}\right]_0^1 = \left(\frac{1^2}{2} + \frac{1^4}{4} + \frac{1^8}{4}\right) - \left(\frac{0^2}{2} + \frac{0^4}{4} + \frac{0^8}{8}\right) =$$
$$= \frac{2+1+1}{4} = 1$$

Příklad 1 (b)

$$\int_1^2 x \ln x\,\mathrm{d}x \overset{\mathrm{PP}}{=} \left|u = \log x,\ v' = x\right| = \left[\frac{x^2}{2}\log x\right]_1^2 - \int_1^2 \frac{x^2}{2}\frac{1}{x}\,\mathrm{d}x = 2\log 2 - 0 - \int_1^2 \frac{x}{2}\,\mathrm{d}x =$$
$$= \log 4 - \left[\frac{x^2}{4}\right]_1^2 = \log 4 - \frac{2^2}{4} + \frac{1}{4} = \log 4 - \frac{3}{4}$$

Příklad 1 (c)

$$\int_0^\pi x^2 \sin x\,\mathrm{d}x \overset{\mathrm{PP}}{=} \left|u = x^2,\ v' = \sin x\right| = \left[-x^2\cos x\right]_0^\pi + \int_0^\pi 2x\cos x\,\mathrm{d}x =$$
$$\overset{\mathrm{PP}}{=} \left|2u = x,\ x' = \cos x\right| = -\pi^2(-1) + 0 + [2x\sin x]_0^\pi - \int_0^\pi 2\sin x\,\mathrm{d}x =$$
$$= \pi^2 + 2\pi\cdot 0 - 0 - 2\left[-\cos x\right]_0^\pi = \pi^2 - 2(1-(-1)) = \pi^2 - 4$$

Příklad 1 (d) $\int\limits_2^3 \frac{x^2-x+1}{x-1}\,\mathrm{d}x$

$$\int f(x)\,\mathrm{d}x = \int_2^3 x + \frac{1}{x-1}\,\mathrm{d}x = \left[\frac{x^2}{2} + \log|x-1|\right]_2^3 = \frac{9}{2} + \log 2 - \frac{4}{2} - \log 1 = \frac{5}{2} + \log 2$$

Příklad 1 (e) $\int\limits_0^\pi \frac{\sin x}{\cos^2 x+1}\,\mathrm{d}x$

Pomocí rozboru fce R z teorie ke goniometrické substituci docházíme k volbě $y = \cos x$

$$\int f(x)\,\mathrm{d}x = |y = \cos x,\ \ \mathrm{d}y = -\sin x\,\mathrm{d}x,\ 0 \mapsto \cos 0 = 1,\ \pi \mapsto \cos(\pi) = -1| = -\int_1^{-1} \frac{1}{1+y^2}\,\mathrm{d}y =$$
$$= \int_{-1}^1 \frac{1}{1+y^2}\,\mathrm{d}y = [\arctan y]_{-1}^1 = \arctan 1 - \arctan(-1) = \frac{\pi}{4} - \frac{-\pi}{4} = \frac{\pi}{2}$$

Příklad 1 (f) $\int\limits_0^1 \sqrt{1+\sqrt{x}}\,\mathrm{d}x$

$$\int f(x)\,\mathrm{d}x = \left| y = 1+\sqrt{x},\ \ \mathrm{d}y = \frac{1}{2\sqrt{x}}\,\mathrm{d}x \implies 2(y-1)\,\mathrm{d}y = \mathrm{d}y,\ 0 \mapsto 1+\sqrt{0},\ 1 \mapsto 1+\sqrt{1} \right| =$$

$$= \int_1^2 \sqrt{y}2(y-1)\,\mathrm{d}y = 2\int_1^2 y^{\frac{3}{2}} - y^{\frac{1}{2}}\,\mathrm{d}y = 2\left[\frac{2y^{\frac{5}{2}}}{5} - \frac{2y^{\frac{3}{2}}}{3}\right]_1^2 =$$

$$= 2\left(\frac{2\sqrt{2^5}}{5} - \frac{2\sqrt{2^3}}{3}\right) - 2\left(\frac{2}{5} - \frac{2}{3}\right) = \frac{16\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} + \frac{8}{15} = \frac{8+8\sqrt{2}}{15}$$

Příklad 2 (a) $\int_0^1 \frac{1}{x^a}\,\mathrm{d}x,\ a \in \mathbb{R}$

$$\int_0^1 x^a\,\mathrm{d}x = \begin{cases} \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}, & a > -1 \\ [\log x]_0^1 = 0 - \lim_{x\to 0+} \log x = \infty, & a = -1 \\ \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1} - \lim_{x\to 0+} \frac{x^{a+1}}{a+1} = \infty, & a < -1 \end{cases}$$

Z výpočtu je vidno, že $\int_0^1 x^a\,\mathrm{d}x = \frac{1}{a+1}$ pro $a > -1$ a pro zbylá a zadaný integrál diverguje.

Příklad 2 (b) $\int_1^\infty \frac{1}{x^a}\,\mathrm{d}x,\ a \in \mathbb{R}$

$$\int_1^\infty x^a\,\mathrm{d}x = \begin{cases} \left[\frac{x^{a+1}}{a+1}\right]_1^\infty = \lim_{x\to\infty} \frac{x^{a+1}}{a+1} - \frac{1}{a+1} = \infty, & a > -1 \\ [\log x]_1^\infty = \lim_{x\to\infty} \log x - 0 = \infty, & a = -1 \\ \left[\frac{x^{a+1}}{a+1}\right]_1^\infty = \lim_{x\to\infty} \frac{x^{a+1}}{a+1} - \frac{1}{a+1} = -\frac{1}{a+1}, & a < -1 \end{cases}$$

Z výpočtu je vidno, že $\int_1^\infty x^a\,\mathrm{d}x = \frac{-1}{a+1}$ pro $a < -1$ a pro zbylá a zadaný integrál diverguje.

Příklad 2 (c) $\int\limits_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$

$$\int\limits_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \left| y = \sqrt{x},\ \ \mathrm{d}y = \frac{1}{2\sqrt{x}}\,\mathrm{d}x,\ 1 \mapsto 1,\ \infty \mapsto \infty \right| = \int_1^\infty 2e^{-y}\,\mathrm{d}y =$$

$$= 2\left[-e^{-y}\right]_1^\infty = 2\left(\lim_{y\to\infty}(-e^{-y}) - (-e^{-1})\right) = 2\left((-0) + \frac{1}{e}\right) = \frac{2}{e}$$

Příklad 2 (d) $\int\limits_4^\infty \frac{1}{x^2}\sqrt{\frac{x-2}{x-4}}\,\mathrm{d}x$

$$\int\limits_4^\infty \frac{1}{x^2}\sqrt{\frac{x-2}{x-4}}\,\mathrm{d}x =$$

$$= \left| y = \sqrt{\frac{x-2}{x-4}},\ -y\left(\frac{4y^2-2}{y^2-1} - 4\right)^2 \mathrm{d}y = \mathrm{d}x,\ 4 \mapsto \lim_{x\to 4+} \sqrt{\frac{x-2}{x-4}} = \infty,\ \infty \mapsto \lim_{x\to\infty} \sqrt{\frac{x-2}{x-4}} = 1 \right| =$$

$$= -\int_\infty^4 \frac{4y^2}{(4y^2-2)^2}\,\mathrm{d}y = \int_4^\infty \frac{4y^2}{(4y^2-2)^2}\,\mathrm{d}y$$

A dopočteme parciální zlomky atd.

Příklad 2 (g) $\int_3^\infty \frac{x-1}{x^2+2x}\,\mathrm{d}x$

$$\int_3^\infty \frac{x-1}{x^2+2x}\,\mathrm{d}x = \left|y = x^2+2x,\ \ \mathrm{d}y = (2x+2)\,\mathrm{d}x,\ 3\mapsto 3^2+2\cdot 3 = 15,\ \infty\mapsto\infty\right| =$$
$$= \int_{15}^\infty \frac{1}{2}\frac{1}{y}\,\mathrm{d}y - 2\int_3^\infty \frac{1}{(x+1)^2-1}\,\mathrm{d}x = |z = x+1,\ \ \mathrm{d}z = \mathrm{d}x,\ 3\mapsto 4,\ \infty\mapsto\infty| =$$
$$= \frac{1}{2}\left[\log|y|\right]_{15}^\infty - 2\int_4^\infty \frac{1}{z^2-1}\,\mathrm{d}z = \frac{1}{2}\left[\log|y|\right]_{15}^\infty + \frac{1}{2}\left[\log\left|\frac{x-1}{x+1}\right|\right]_4^\infty =$$
$$= \frac{1}{2}(\infty - \log 15) + \frac{1}{2}\left(\log 1 - \log\frac{3}{5}\right) = \infty$$

Příklad 3 (a) $\int_1^2 \frac{3x^2}{x^3+1}\,\mathrm{d}x$

$$\int_1^2 \frac{3x^2}{x^3+1}\,\mathrm{d}x \overset{\text{par. zlomky}}{=} \int_1^2 \frac{2x-1}{x^2-x+1} + \frac{1}{x+1}\,\mathrm{d}x =$$
$$= \left|y = x^2-x+1,\ \ \mathrm{d}y = (2x-1)\,\mathrm{d}x,\ 1\mapsto 1,\ 2\mapsto 3\right| =$$
$$= \int_1^3 \frac{1}{y}\,\mathrm{d}y + \left[\log|x+1|\right]_1^2 = \left[\log|y|\right]_1^3 + \log 3 - \log 2 = \log 3 - \log 1 + \log 3 - \log 2 = \log\frac{9}{2}$$

Příklad 3 (b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x\cos x\,\mathrm{d}x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x\cos x\,\mathrm{d}x \overset{\text{PP}}{=} \left|u = \sin x,\ v' = \cos x\right| = \left[\sin^2 x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x\cos x\,\mathrm{d}x$$

$$\Longrightarrow\ 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x\cos x\,\mathrm{d}x = \left[\sin^2 x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x\cos x\,\mathrm{d}x = \frac{1}{2}\left(\sin^2\frac{\pi}{3} - \sin^2\frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{1}{8}$$

Příklad 3 (c) $\int_1^e \frac{\log^2 x}{x}\,\mathrm{d}x$

$$\int_1^e \frac{\log^2 x}{x}\,\mathrm{d}x = \left|y = \log x,\ \ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x,\ 1\mapsto 0,\ e\mapsto 1\right| = \int_0^1 y^2\,\mathrm{d}y = \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}$$

Příklad 3 (d) $\int_{-1}^1 \frac{x^2}{1+x^2}\,\mathrm{d}x$

$$\int_{-1}^1 \frac{x^2+1-1}{1+x^2}\,\mathrm{d}x = \int_{-1}^1 1 - \frac{1}{1+x^2}\,\mathrm{d}x = \left[x - \arctan x\right]_{-1}^1 =$$
$$= 1 - \arctan 1 - (-1) + \arctan(-1) = 2 - \frac{\pi}{4} + \frac{-\pi}{4} = 2 - \frac{\pi}{2}$$

Příklad 3 (e) $\int_0^\infty \frac{1}{(x+3)^5}\,\mathrm{d}x$

$$\int_0^\infty \frac{1}{(x+3)^5}\,\mathrm{d}x = |y = x+3,\;\; \mathrm{d}y = \mathrm{d}x,\; 0 \mapsto 3,\; \infty \mapsto \infty| = \int_3^\infty \frac{1}{y^5}\,\mathrm{d}y = \left[-\frac{1}{4y^4}\right]_3^\infty =$$
$$= -\lim_{y\to\infty} \frac{1}{4y^4} + \frac{1}{4\cdot 3^4} = \frac{1}{324}$$

Příklad 3 (f) $\int_0^1 \frac{e^x}{e^{2x}+1} + \frac{1}{\cos^2 x}\,\mathrm{d}x$

$$\int_0^1 \frac{e^x}{e^{2x}+1} + \frac{1}{\cos^2 x}\,\mathrm{d}x = |y = e^x,\;\; \mathrm{d}y = e^x,\; 0 \mapsto 1,\; 1 \mapsto e| = \int_1^e \frac{1}{y^2+1}\,\mathrm{d}y + [\tan x]_0^1 =$$
$$= \tan 1 + [\arctan y]_1^e = \tan 1 + \arctan e - \frac{\pi}{4}$$

Příklad 3 (g) $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \left| y = \sqrt{x},\;\; \mathrm{d}y = \frac{1}{2\sqrt{x}}\,\mathrm{d}x,\; 1 \mapsto 1,\; \infty \mapsto \infty \right| = \int_1^\infty 2e^{-y}\,\mathrm{d}y = \left[-2e^{-y}\right]_1^\infty =$$
$$= \lim_{y\to\infty} \left(-2e^{-y}\right) + 2e^{-1} = \frac{2}{e}$$